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SUMMARY

ENGINEERING RELIABILITY

FUNDAMENTALS OF PROBABILITY

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Outline

Preliminary Definitions

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Random Variable

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Summary

- A random variable (denoted by X) is a variable that can assume one or more possible numerical values (denoted by x)
- The value x that X assumes is determined by chance
- A random variable may be:
  - discrete

$$x \in \{1, 2, 3\}$$

  - or continuous

$$x \in \{x \,|\, 0 \leq x < \infty\}$$

Probability Distribution/Probability Density

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- for discrete random variables the probability distribution function is the set of probabilites

$$f(x) = P(X = x), \quad \text{and} \quad \sum_x f(x) = 1$$

- for continuous random variables the (cumulative) probability function is the function:

$$F(x) = P(X \leq x)$$

- for continuous random variables the probability distribution or probability density is:

$$f(x) = \frac{d}{dx} F(x), \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Probability of Failure/Reliability

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- Let the random variable T denote the time of failure of a product during service. Then the possible values of T are the set $0 < t < \infty$.
- The associated probability density is $f(t)$, the probability that failure occurs at time t is

$$F(t) = \int_0^t f(\tau)\, d\tau$$

- The reliability function is the probability of survival to time t

$$R(t) = P(T \geq t) = 1 - F(t) = \int_t^\infty f(\tau)\, d\tau$$

Sample Space

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The underlying idea is that there is a well-defined trial and a set of possible outcomes.

Example (Flipping a coin 3 times)

Flipping a coin 3 times yields the following set of outcomes:

$$\begin{array}{llll} \xi_1 = TTT & \xi_2 = TTH & \xi_3 = THT & \xi_4 = THH \\ \xi_5 = HTT & \xi_6 = HTH & \xi_7 = HHT & \xi_8 = HHH \end{array}$$

- Sample space, S, the set of all possible outcomes.
- Elementary outcome, ξ, the individual elements of S.

Events

Event A: a subset of the sample space. In general, an event is defined by a proposition about the elements in it.

Example (Flipping a coin 3 times)

A is the event that a tail shows up on the second toss

$$A = \{\xi_1, \xi_2, \xi_5, \xi_6\}$$

Elementary event: $\{\xi\}$, where ξ is an elementary outcome.
Sure event: the entire sample space, S.
Impossible event: the empty set, $\emptyset$.
Complementary event: A^c consists of all events not in A.
Mutually Exclusive events: events that have pairwise empty intersections.

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Probability

A probability measure is a function that assigns a 'likelihood' of occurrence to each subset of S (to each event)

A Probability Measure P is a function on the set of subsets of S that has the following properties:

- $P(S) = 1$
- $P(A) \geq 0$ for each $A \subset S$
- For any sequence of mutually exclusive events $A_1, A_2, \ldots$

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Venn Diagrams

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The Venn diagram is an aid in visualizing basic properties of sets. It can also serve as a simple visual representation of the probability model.

- The sample space S is visualized as a rectangular set of points in the plane.
- The elementary outcomes are the points in S.
- The probability is associated with a the (non-uniform) distribution of a unit mass over the set S. In the case of a finite number of outcomes, the mass is concentrated at a finite number of points.

VENN DIAGRAMS

BASIC SET OPERATIONS

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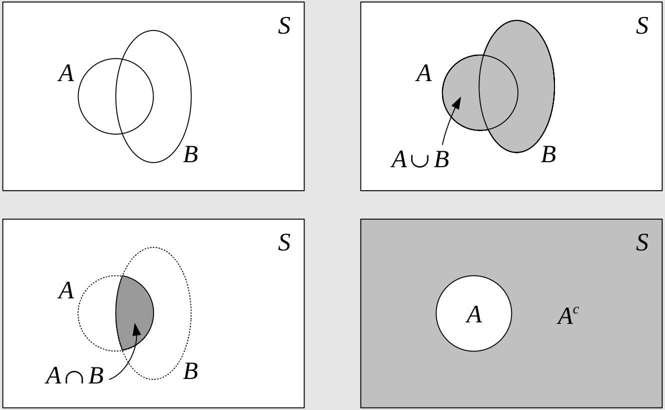

Conditional probability & Independent Events

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The probability that an event A occurs, given the occurrence of an event B is called the conditional probability of A given B. It is denoted $P(A|B)$. From the Venn diagram we see that,

$$P(A \cap B) = P(A|B)\,P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \textit{if } P(B) \neq 0$$

Two events A and B are independent if the occurrence of one does not 'condition' the occurrence of the other, i.e.,

$$P(A|B) = P(A) \quad \textit{and} \quad P(B|A) = P(B)$$

Thus, for independent events

$$P(A \cap B) = P(A)\,P(B)$$

INDEPENDENT EVENTS

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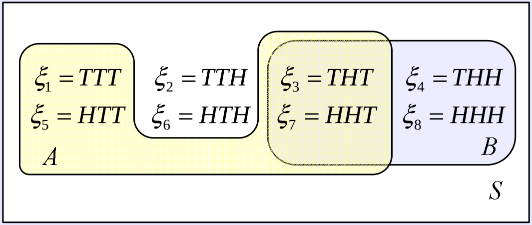
EXAMPLE (FLIPPING A COIN 3 TIMES)

- reconsider the coin flipping experiment with sample space S shown below.
- let A be the event that a T occurs on the third toss.
- let B be the event that an H occurs on the second toss.

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(A|B) = \frac{1}{2}, \quad P(B|A) = \frac{1}{2}$$

Notice that the events are independent, but not mutually exclusive.



Bayes' Rule

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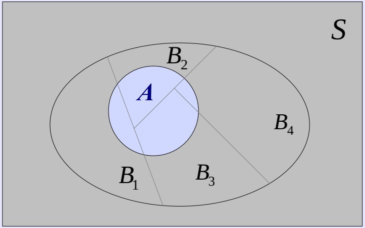

- expansion rule: $A \subset \cup_{i \in J} B_i$, events B_i mutually exclusive

$$P(A) = \sum_{i \in J} P(A|B_i)\, P(B_i)$$

- Bayes' rule: $A \subset \cup_{i \in J} B_i$, events B_i mutually exclusive

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i)\, P(B_i)}{\sum_{i \in J} P(A|B_i)\, P(B_i)}$$

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Summary

A *population of size* n is a set of n distinguishable elements.

Consider a population of size n from which we obtain an 'ordered' sample of size r.

- Sampling with replacement there are n^r possible samples.
- Sampling without replacement there are $(n)_r = n(n-1)\cdots(n-r+1)$ possible samples.
- Set $n = r$ to find that there are $n!$ different orderings of n elements.

Binomial Coefficients

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We want to choose a subpopulation of size r from a population of size n. How many different such subpopulations are there?

- There are $(n)_r$ samples of size r without replacement,
- Each r-sample can be ordered in $r!$ ways,
- Thus, there are $(n)_r/r!$ subpopulations of size r.

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n!}{r!(n-r)!} = C_r^n$$

Bernoulli Trials

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- By a Bernoulli trial we mean an experiment consisting of a sequence of independent trials with two possible outcomes, *success* and *failure*, having probability of failure p and probability of success $q = 1 - p$.
- *Fundamental Problem*: Consider a Bernoulli trial of Length n. What is the probability of exactly k failures?
- The event 'n trials results in k failures and $n - k$ successes' can happen in as many ways as k letters F can be distributed among n places.
- In other words, i.e., how many subpopulations of size k can be constructed from a population of size n? The event consists of:

$$C_k^n \text{ points, each with probability } p^k q^{n-k}$$

BINOMIAL DISTRIBUTION

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The probability that n Bernoulli trials with probability p for failure and $q = 1 - p$ for success results in k failures and $n - k$ successes is given by the *Binomial distribution*:

$$b\left(k; n, p\right) = \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}, \; 0 \leq k \leq n$$

Poisson Approximation

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If

- the probability of failure p is small,
- the number of trials n is large, so that $np = \lambda$, a constant

then a good approximation to the Binomial distribution is the *Poisson distribution*:

$$p(k;\lambda) = e^{-\lambda}\frac{\lambda^k}{k!}, \;\; \lambda = np$$

Random Variables

A function $X(\cdot)$ that maps a sample space S to the real numbers R is called a (real valued) random variable if it has the property:

$$\{\xi \in S | X(\xi) \leq x\} \text{ is an event } \forall x \in R$$

All elementary outcomes that result in $X(\xi) \leq x$ is a valid subset of S, for all real x.

- A random variable is discrete if it can assume a finite set of distinct values, say $x_i, i = 1, \ldots, n < \infty$
- A random variable is continuous if the values it can assume are continuously distributed over its range, say $-\infty < x < \infty$

Discrete random Variables

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- $f\left(x_i\right) = P\left(X = x_i\right)$ is called the probability distribution, note: $\sum_i f\left(x_i\right) = 1$
- the (cumulative) probability function is:
  $F\left(x_k\right) = \sum_{i=1}^{k} f\left(x_i\right)$
  note: $F\left(x_k\right) = P\left(X \leq x_k\right)$
- mean: $\mu = \sum_{i=1}^{n} x_i f(x_i)$
- variance: $\sigma^2 = \sum_{i=1}^{n} \left(x_i - \mu\right)^2 f(x_i)$
- standard deviation: σ

Coin Flipping Example

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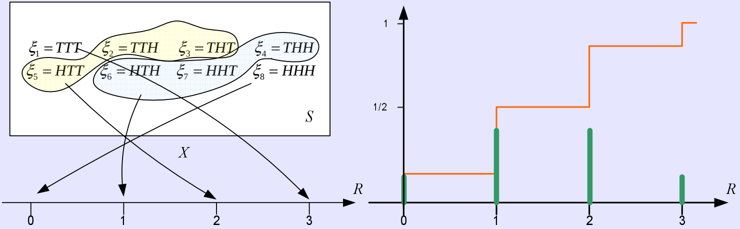
Example (Flipping a coin 3 times)

- Flipping a coin 3 times yields a set of 8 outcomes.
- Assume: on a single toss, $P(H) = P(T) = 1/2$.
- Define: X =number of tails in 3 tosses.



Continuous Random Variables

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Summary

- the (cumulative) probability function is: $F(x) = P(X \leq x)$
- the probability distribution (density) is:

$$f(x) = \frac{d}{dx}F(x) \Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$$

- mean: $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$
- variance: $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$
- standard deviation: σ

Continuous Random variables, cont'd

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- median: $x_m, \ F(x_m) = \int_{-\infty}^{x_m} f(x)\,dx = \frac{1}{2}$
- mode: $x_{\text{mode}}, \ f(x_{\text{mode}}) \geq f(x)$
- skewness: $sk = \frac{1}{\sigma^3} \int_{-\infty}^{\infty} (x-\mu)^3 f(x)\,dx$

Comments on skewness:

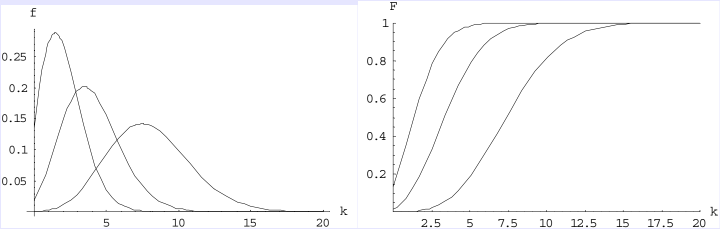
- $sk > 0 \Rightarrow \text{left} - \text{skewed} : x_{\text{mode}} < x_m < \mu$
- $sk < 0 \Rightarrow \text{right} - \text{skewed} : x_{\text{mode}} > x_m > \mu$
- $sk = 0 \Rightarrow \text{symmetric} : x_{\text{mode}} = x_m = \mu$

Example – Poisson Distribution

$$f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$F(k) = \sum_{i=1}^{k} e^{-\lambda} \frac{\lambda^i}{i!} = \frac{(1+k)\,\Gamma(1+k,\lambda)}{\Gamma(2+k)}$$

Here is the Poisson distribution for $\lambda = 2, 4, 8$ (left to right)



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Example – Poisson Distribution, Cont'd

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$$\mu = \sum_{k=0}^{\infty} k \left(e^{-\lambda} \frac{\lambda^k}{k!} \right) = \lambda$$

$$\sigma^2 = \sum_{k=0}^{\infty} (k - \mu)^2 \left(e^{-\lambda} \frac{\lambda^k}{k!} \right) = \lambda$$

$$sk = \frac{1}{\sigma^3} \sum_{k=0}^{\infty} (k - \mu)^3 \left(e^{-\lambda} \frac{\lambda^k}{k!} \right) = \frac{1}{\sqrt{\lambda}}$$

THE NORMAL DISTRIBUTION

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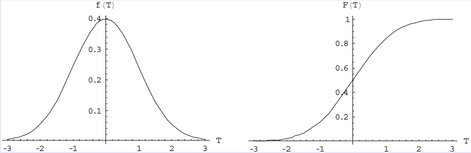
The normal distribution has two key applications in reliability:

- It is a good model for the variability of parameters in batch-manufactured parts.
- It is a good approximation to the 'wear-out' time to failure distribution.

$$f\left(T\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T-\mu}{\sigma}\right)^2}$$

where μ is the mean time to failure and σ is the standard deviation of the time to failure.



Standard Normal Distribution

The normal distribution is commonly used that the standard normal distribution has been introduced to facilitate computations. Suppose X is a normally distributed random variable with mean μ and standard deviation σ. Consider a new random variable Z, related to X, by the relation

$$Z = (X - \mu)/\sigma$$

Clearly,

$$P(X \leq x) = P(Z \leq (x - \mu)/\sigma)$$

Equivalently,

$$F(x) = \Phi((x - \mu)/\sigma)$$

where $F(x)$ is the probability function for X and $\Phi(z)$ is that of Z. Thus,

$$f(x) = \frac{dF(x)}{dx} = \frac{d\Phi(z)}{dz}\frac{dz}{dx} = \phi(z)\frac{dz}{dx}$$

Standard Normal Distribution – 2

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It follows that

$$\phi\left(z\right) = \left[f\left(x\right)\left(\frac{dz}{dx}\right)^{-1}\right]_{x\to\sigma z+\mu} = \left[\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\sigma\right]_{x\to\sigma z+\mu}$$

So,

$$\phi\left(z\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

and

$$\Phi\left(z\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}\zeta^2}\,d\zeta$$

The standard normal is the normal distribution wit zero mean and unit variance. It's usefulness follows from the fact that

$$F\left(x\right) = \Phi\left(\left(x-\mu\right)/\sigma\right), \quad f\left(x\right) = \phi\left(\left(x-\mu\right)/\sigma\right)$$

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Example (Flat Panel Monitor)

A flat pane computer monitor is designed to operate for 10,000 hours. Data shows that 2% fail within 1000 hours and 3.8% fail before 2000 hours. Assuming a normal time to failure distribution, determine the mean time to failure.

$$F(t) = \int_{-\infty}^{t} f(\tau) d\tau = \frac{1}{2}\left(1 + \mathrm{Erf}\left[\frac{t-\mu}{\sqrt{2}\,\sigma}\right]\right)$$

We can obtain the unknown parameters μ, σ from the two relations

$$F(1000) = 0.02, F(2000) = 0.038 \Rightarrow \mu = 8351, \sigma = 3580$$

Bayes' Rule – Continuous Random Variables

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Suppose X and Y are random variables. The following quantities are defined:

- joint probability distribution is

$$F(x,y) = P(X \leq x) \cap P(Y \leq y)$$

- joint probability density function is

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

- conditional density function for X given Y

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

Bayes rule for densities

$$f(x|y) = \frac{f(y|x)f(x)}{\int_{-\infty}^{\infty} f(y|x)f(x)\,dx}$$

Summary

- Random variable, probability, reliability
- Sample space, events
- Conditional probability, Bayes' rule
- Combinatorics, Bernoulli trials
- Discrete random variables - Binomial and Poisson distributions
- Continuous random variables - Normal distribution
- Mean, variance skewness, median, mode