Engineering Reliability

Failure Models

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Outline

Introduction

We will be concerned with the following measures of a product or system that is not repaired upon failure.

- The reliability (or survivor) function, $R(t)$.
- The failure rate, $z(t)$ or $\lambda(t)$.
- The mean time to failure, $MTTF$.
- The mean residual life MRL.
- Constant rate models.

The Meaning of Time

The failure time T of a product or systems is a random variable. Time can take on different meanings depending on the context, e.g.,

- Calender time.
- Operational time.
- Distance driven by a vehicle.
- Number of cycles for a periodically operated system.
- Number of times a switch is operated.

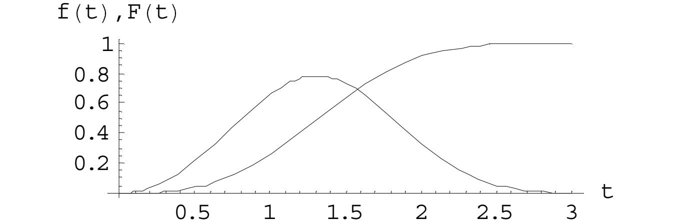
Probability Characterization of Failure Time

Associated with the time to failure T is the probability function

$$F(t) = P(T \leq t)$$

which is the probability that the system fails within the time interval $(0, t]$. If T is a continuous random variable, the probability function is related to its probability density function $f(t)$ by

$$F(t) = \int_0^t f(\tau)\, d\tau$$

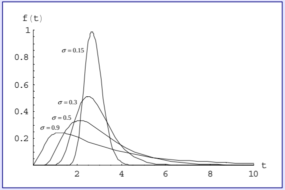


LOGNORMAL DISTRIBUTION

- The lognormal distribution has been found useful in the failure analysis of items subjected to repeated loadings
- while the normal distribution is ideal for characterizing the influence of the sum of a large number of independent events, the lognormal is appropriate for characterizing the product of a large number of independent events

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2}, \quad t > 0$$

$$F(t) = \frac{1}{2}\left(1 + \mathrm{Erf}\left(\frac{\ln t - \mu}{\sqrt{2}\sigma}\right)\right), \quad t > 0$$

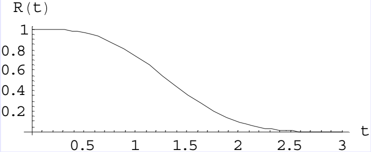


Reliability Function

The reliability function is:

$$R(t) = P(T > t) = 1 - F(t) = \int_t^\infty f(\tau)\, d\tau$$

- $R(t)$ is the probability that the item will not fail in the interval $(0, t]$.
- $R(t)$ is the probability that it will survive at least until time t – it is sometimes called the survival function.



Failure Rate

Consider the conditional probability:

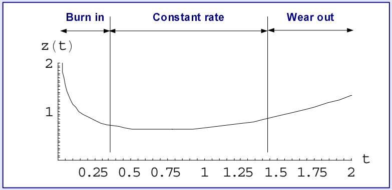
$$P\left(t < T \leq t + \Delta t \,|T > t\right) = \frac{P\left(t < T \leq t + \Delta t\right)}{R\left(t\right)}$$

$$= \frac{F\left(t + \Delta t\right) - F\left(t\right)}{R\left(t\right)}$$

The failure rate (or, hazard function) is defined as:

$$\lambda\left(t\right) = \lim_{\Delta t \to 0} \frac{P\left(t < T \leq t + \Delta t \,|T > t\right)}{\Delta t} = \frac{f\left(t\right)}{R\left(t\right)}$$

$\lambda(t)\ dt$ is the probability that the system will fail during the period $(t, t + dt]$, given that it has survived until time t.



Distributions from Failure Rate

Suppose the failure rate $\lambda(t)$ is known. Then it is possible to obtain $f(t)$, $F(t)$, and $R(t)$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \Rightarrow \lambda(t) = -\frac{dR/dt}{R}$$

$$\frac{dR}{R} = -\lambda(t)\,dt$$

$$\Downarrow$$

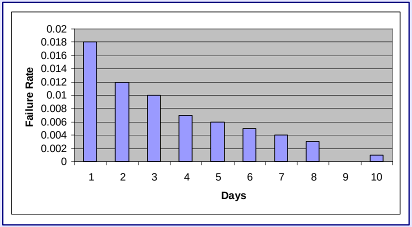
$$\boxed{\begin{array}{c} R(t) = \exp\left[-\int_0^t \lambda(\tau)\,d\tau\right] \\ f(t) = \lambda(t)\exp\left[-\int_0^t \lambda(\tau)\,d\tau\right] \\ F(t) = 1 - \exp\left[-\int_0^t \lambda(\tau)\,d\tau\right] \end{array}}$$

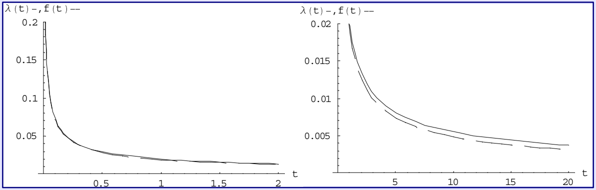
Example – TV Sets

Example (TV Set Failure Data)

Day	failures	failure rate
1	18	0.018
2	12	0.012
3	10	0.010
4	7	0.007
5	6	0.006
6	5	0.005
7	4	0.004
8	3	0.003
9	0	0
10	1	0.001

Example – TV Sets, Cont'd





$$\lambda(t) = 0.02t^{-0.56} \quad f(t) = 0.02t^{-0.56}e^{-0.04545t^{0.44}}$$

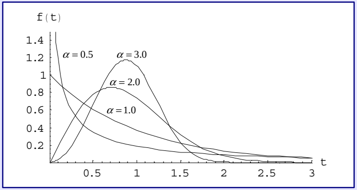
WEIBULL DISTRIBUTION

Perhaps the most frequently used distribution to model time to failure probabilities is the Weibull distribution, Weibull(α, β), The probability function is

$$F(t) = \begin{cases} 1 - e^{-(\beta t)^{\alpha}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and the corresponding density function is

$$f(t) = \frac{d}{dt} F(t) = \begin{cases} \alpha \beta^{\alpha} t^{\alpha - 1} e^{-(\beta t)^{\alpha}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Weibull Distribution – (2)

Engineering Reliability

Introduction

Failure Rate

Time to Failure

Reliability Function

Failure Rate

MTTF & MRL

Constant Rate Models

The Exponential Distribution

Repeated Demand

Variable Rate Models

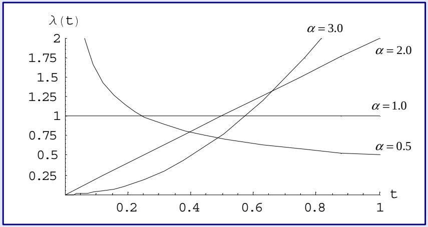
Summary

The Reliability function is:

$$R(t) = 1 - F(t) = e^{-(\beta t)^{\alpha}}, \quad t > 0$$

and the failure rate function is

$$\lambda(t) = \frac{f(t)}{R(t)} = \alpha \beta^{\alpha} t^{\alpha - 1}, \quad t > 0$$



Mean Time to Failure

The mean time to failure is

$$MTTF = E\{T\} = \int_0^\infty tf(t)\,dt$$

note that

$$f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}R(t)$$

from which

$$MTTF = -\int_0^\infty t\frac{dR(t)}{dt}\,dt = -\left. tR(t)\right|_0^\infty + \int_0^\infty R(t)\,dt$$

$$\boxed{MTTF = \int_0^\infty R(t)\,dt}$$

EXAMPLE

EXAMPLE

Consider a system with reliability function

$$R(t) = \frac{1}{(0.2t+1)^2}, \text{ for } t > 0 \quad (t \text{ in months})$$

- probability density $f(t) = -\frac{d}{dt}R(t) = \frac{0.4}{(0.2t+1)^3}$
- failure rate $\lambda(t) = \frac{f(t)}{R(t)} = \frac{0.4}{(0.2t+1)}$
- mean time to failure $MTTF = \int_0^\infty R(t)\, dt = 5$ months

Mean Residual Life

An item begins operation at time 0 and is still operating at time t. We wish to compute the probability that it will survive an additional interval of length τ. This is the conditional reliability function at age t.

$$R\left(\tau\,|t\right) = P\left(T > t + \tau\,|T > t\right) = \frac{P\left(T > t + \tau\right)}{P\left(T > t\right)} = \frac{R\left(t + \tau\right)}{R\left(t\right)}$$

The mean residual (or, remaining) life at age t is

$$MRL\left(t\right) = \int_0^\infty R\left(\tau\,|t\right)\,d\tau = \frac{1}{R\left(t\right)}\int_t^\infty R\left(\tau\right)d\tau$$

Example

Consider an item with failure rate $\lambda\left(t\right) = t/\left(t+1\right)$ Now compute

$$R\left(t\right) = \exp\left(-\int_0^t \frac{\tau}{\tau+1}\,d\tau\right) = \left(t+1\right)e^{-t}$$

$$MTTF = \int_0^\infty \left(t+1\right)e^{-t}\,dt = 2$$

The conditional reliability function is

$$R\left(\tau\,|t\right) = P\left(T > \tau + t\,|T > \tau\right) = \frac{\left(t+\tau+1\right)e^{-\left(t+\tau\right)}}{\left(t+1\right)e^{-t}} = \frac{t+\tau+1}{t+1}$$

So,

$$MRL = \int_0^\infty R\left(\tau\,|t\right)d\tau = 1 + \frac{1}{t+1}$$

Example: Weibull Distribution

The mean time to failure is

$$MTTF = \int_0^\infty R(t)\,dt = \frac{1}{\beta}\Gamma\left(\frac{1}{\alpha}+1\right)$$

The remaining life is

$$MRL = \frac{1}{R(t)}\int_t^\infty R(t)\,dt = \left(\frac{\alpha}{\beta}\right)e^{\left(\frac{t}{\beta}\right)^\alpha}\Gamma\left(\frac{1}{\alpha},\left(\frac{t}{\beta}\right)^\alpha\right)$$

The median life is

$$R(t_m) = 0.50 \Rightarrow t_m = \frac{1}{\beta}(\ln 2)^{1/\alpha}$$

The variance of T is

$$\operatorname{var}(T) = \frac{1}{\beta^2}\left(\Gamma\left(\frac{2}{\alpha}+1\right)+\Gamma^2\left(\frac{1}{\alpha}+1\right)\right)$$

The Exponential Distribution

Suppose $\lambda(t) = \lambda_0$, a constant. Then,

$$R(t) = e^{-\lambda_0 t}, \quad F(t) = 1 - e^{-\lambda_0 t}, \quad f(t) = \lambda_0 e^{-\lambda_0 t}$$

This is the exponential distribution. We can easily compute

$$\mu = MTTF = \frac{1}{\lambda_0}$$

$$\sigma = \frac{1}{\lambda_0}$$

$$R(\mu) = 0.368, \quad F(\mu) = 0.632$$

Example

Suppose $\lambda = .02\ hr^{-1}$,

- What is the probability of a failure in the first 10 hours of service?

$$P\left(T \leqslant 10\right) = F\left(10\right) = 1 - e^{0.02\times10} = 0.181$$

- Suppose the unit operates satisfactorily for the first 100 hours. What is the probability of failure in the next 10 hours?
  - Let $X =$ event that the unit operates for 100 hours $\Rightarrow P\left(X\right) = R\left(100\right) = .135$
  - Let $Y =$ event that the unit fails within 110 hours $\Rightarrow P\left(Y\right) = F\left(110\right) = .1108$
  - We want to compute $P\left(Y|X\right)$. By definition

$$P\left(Y|X\right) = \frac{P\left(Y \cap X\right)}{P\left(X\right)} = \frac{P\left(100 \leqslant t \leqslant 110\right)}{P\left(X\right)} = \frac{F\left(110\right) - F\left(100\right)}{R\left(100\right)} = 0.181$$

Unit Under Repeated Demand

Suppose a unit is subjected to repeated demand (e.g., engine startup) over a large time interval $(0, t]$:

- p denotes the probability of failure to respond to a demand,
- the response to each demand is an independent event,
- n denotes the number of demands in time t,
- $m = n/t$ denotes the average number of demands per unit time,

The probability of k failures in n demands is given by the Binomial distribution $b(k; n, p) = C_k{}^n p^k (1-p)^{n-k}$.
Let N denote the number of trials until the first failure. Thus, $N = n$ means that the first $n-1$ trials are successful, and failure occurs at trial n. The distribution of N is the geometric distribution

$$P(N = n) = (1-p)^{n-1} p$$

UNIT UNDER REPEATED DEMAND, CONT'D

The reliability function, $R(n)$, is the probability that the first failure occurs for some trial $N > n$. Consequently,

$$R(n) = P(N > n) = b(0; n, p) = (1-p)^n$$

Suppose the single trial failure probability p is small and the length of the observed sequence N large, in fact $Np = \lambda$, $\lambda = \text{constant}$. Then,

$$\lim_{p \to 0} (1-p)^{\frac{\lambda}{p}} = e^{-\lambda} \Rightarrow R(n) = e^{-np} = e^{-mpt} = e^{-\lambda_0 t}$$

where $\lambda_0 = mp$ is the equivalent failure rate.

Multiple Performance Levels

A unit operates at different performance levels during cyclic operation. In each operating phase the unit fails at constant rate.

Example

- a motor cycles through 3 phases: start, run, standby,
- N, number of starts per service cycle,
- T denotes the number of service cycles to failure,
- c, time fraction motor runs during a cycle,
- $1 - c$, time fraction motor is in standby,
- p, probability of failure to start,
- λ_r, failure rate in run state,
- λ_s, failure rate in standby state.

$$\lambda_c \triangleq \lambda_d + c\lambda_r + (1-c)\,\lambda_s, \text{ where } \lambda_d \triangleq Np$$

$$R(t) = e^{-\lambda_c t}$$

Common Variable Rate Models

- Time to failure is typically described by normal, lognormal or Weibull probability distributions.
- The corresponding failure rates can be computed from

$$\lambda(t) = -\frac{1}{R(t)}\frac{dR(t)}{dt}$$

$$Normal: \; \lambda(t) = \frac{\phi(z)}{\sigma(1-\Phi(z))}; \quad z = \frac{t-\mu}{\sigma}$$

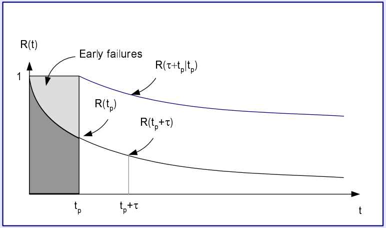
$$Lognormal: \; \lambda(t) = \frac{\phi(z)}{\sigma t \Phi(z)}; \quad z = \frac{\ln(t)-\mu}{\sigma}$$

$$Weibull: \; \lambda(t) = (\alpha\beta)(\beta t)^{\alpha-1}$$

The Wear-in Failure Mode & the Proof Test

- When initial failure rates are high, testing can improve the reliability of deployed product.
- An initial, short period, t_p, of testing of the entire batch weeds out faulty product.
- suppose $\lambda(t) = at^{-b}$, $a, b > 0$, the we can compute $R(t - t_p | t_p)$

$$R(\tau | t_p) = \frac{R(t_p + \tau)}{R(t_p)} = \frac{e^{-\int_0^{t_p+\tau} \lambda(\xi) d\xi}}{e^{-\int_0^{t_p} \lambda(\xi) d\xi}} = e^{-\int_{t_p}^{t_p+\tau} \lambda(\xi) d\xi}$$



Summary

- Time to failure
- Failure rate
- Computation of probability functions from failure rate
- definitions of mean time to failure ($MTTF$) and remaining life (MRL)
- Introduced lognormal, exponential and Weibull distributions
- Examples of constant failure rate problems
- Proof Testing